Inequality 1

1. Prove the inequalty $a^2+b^2 \geq 2ab$. If $x+y+z=c$, show that $x^2+y^2+z^2 \geq \frac{1}{3}c^2$.

$(a-b)^2 \geq 0 \Leftrightarrow a^2+b^2 \geq 2ab$

$$3(x^2+y^2+z^2) = x^2+y^2+z^2+(x^2+y^2)+(y^2+z^2)+(z^2+x^2)$$
$$\geq x^2+y^2+z^2+2xy+2yz+2zx = (x+y+z)^2 = c^2$$

$\therefore x^2+y^2+z^2 \geq \frac{1}{3}c^2$

2. Find the solution set for the inequlaity: $\left|\frac{4}{x-1}\right| \geq 3-\frac{3}{x}$

Note that $x \neq 0,1$.

Case 1 If $x < 1$,

(a) If $x > 0$, that is $0 < x < 1$

$$-\frac{4}{x-1} \geq 3\left(\frac{x-1}{x}\right) \Longrightarrow \frac{4}{1-x} \geq 3\left(\frac{x-1}{x}\right) \Longrightarrow 4x \geq -3(1-x)^2 \Longrightarrow 3x^2-2x+3 \geq 0$$

Since $\Delta = (-2)^2-4(3)(3) < 0$, $3x^2-2x+3 \geq 0$ is always true.

$0 < x < 1$ is a solution.

(b) If $x < 0$,

$$-\frac{4}{x-1} \geq 3\left(\frac{x-1}{x}\right) \Longrightarrow \frac{4}{1-x} \geq 3\left(\frac{x-1}{x}\right) \Longrightarrow 4x \leq -3(1-x)^2 \Longrightarrow 3x^2-2x+3 \geq 0$$

Since $\Delta = (-2)^2-4(3)(3) < 0$, $3x^2-2x+3 \leq 0$ is always false.

There is no solution in this case.

Case 2 If $x > 1$,

$$\frac{4}{x-1} \geq 3\left(\frac{x-1}{x}\right) \Longrightarrow 4x \geq 3(x-1)^2 \Longrightarrow 3x^2-10x+3 \leq 0 \Longrightarrow (3x-1)(x-3) \leq 0$$

$$\Longrightarrow \frac{1}{3} \leq x \leq 3$$

Together with $x > 1$, the solution in this case is $1 < x \leq 3$.

Combining case 1 and 2, the solution is $0 < x < 1$ and $1 < x \leq 3$.

3. Solve $|5-2x| \leq 3x+10$, $|5-2x| < 3x+10$

Method 1

Since $|5-2x| \geq 0$, in order the given inequality to have solution, $3x+10 \geq 0 \Longrightarrow x \geq -\frac{10}{3}$

If $x \geq -\frac{10}{3}$, $|5-2x|^2 \leq (3x+10)^2 \Longrightarrow 25-20x+4x^2 \leq 9x^2+60x+100$

$$\Longrightarrow 5x^2+80x+75 \geq 0 \Longrightarrow x^2+16x+15 \geq 0 \Longrightarrow (x+1)(x+15) \geq 0$$
$$\Longrightarrow x \leq -15 \ or \ x \geq -1$$

Since $x \geq -\frac{10}{3}$, the solution is $x \geq -1$.

Method 2

$|5-2x| \leq 3x+10$

$\begin{cases} 5-2x \geq 0 \\ 5-2x \leq 3x+10 \end{cases}$ or $\begin{cases} 5-2x \leq 0 \\ -(5-2x) \leq 3x+10 \end{cases}$

$\begin{cases} x \leq \frac{5}{2} \\ x \geq -1 \end{cases}$ or $\begin{cases} x \geq \frac{5}{2} \\ x \geq -15 \end{cases}$

$-1 \leq x \leq \frac{5}{2}$ or $x \geq \frac{5}{2}$

$\therefore x \geq -1.$

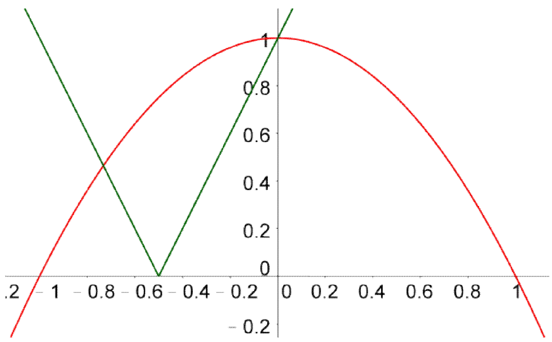
4. Sketch on the same axes, the graphs of $y = |2x+1|$ and $y = 1 - x^2$.
Hence, solve the inequality $|2x+1| \geq 1 - x^2$.

Solve :

$\begin{cases} y = -(2x+1) \\ y = 1 - x^2 \end{cases}$

$\left(x = \sqrt{3}+1, y = -2\sqrt{3}-3\right)$

or $\left(x = 1-\sqrt{3}, y = 2\sqrt{3}-3\right)$



Solution for

$|2x+1| \geq 1 - x^2$

is $x \leq 1-\sqrt{3}$ or $x \geq 0$

5. Show that

$-2 \leq \frac{4x}{4x^2+2x+1} \leq \frac{2}{3}$, where x is real.

Method 1 **(Quadratics)**

Let $y = \frac{4x}{4x^2+2x+1}$

$y(4x^2+2x+1) = 4x$

$(4y)x^2 + (2y-4)x + y = 0$

Since x is real, $\Delta = (2y-4)^2 - 4(4y)y \geq 0$

$16 - 16y - 12y^2 \geq 0$

$3y^2 + 4y - 4 \leq 0$

$(y+2)(3y-2) \leq 0$

$-2 \leq y \leq \frac{2}{3}$

Maximum is $\frac{3}{2}$. Minimum is -2.

Method 2 (Calculus)

Let $y = \frac{4x}{4x^2+2x+1}$

$\frac{dy}{dx} = \frac{(4x^2+2x+1)(4)-4x(8x+2)}{(4x^2+2x+1)^2} = \frac{-4(4x^2-1)}{(4x^2+2x+1)^2}$

For critical values, $\frac{dy}{dx} = 0 \Longrightarrow 4x^2 - 1 = 0 \Longrightarrow x = \pm\frac{1}{2}$

For $x < -\frac{1}{2}, \frac{dy}{dx} < 0$

For $-\frac{1}{2} < x < \frac{1}{2}$, $\frac{dy}{dx} > 0$

For $x > \frac{1}{2}, \frac{dy}{dx} < 0$

When $x = -\frac{1}{2}$, y is a minimum, $y = \frac{4\left(-\frac{1}{2}\right)}{4\left(-\frac{1}{2}\right)^2+2\left(-\frac{1}{2}\right)+1} = -2$

When $x = \frac{1}{2}$, y is a maximum, $y = \frac{4\left(\frac{1}{2}\right)}{4\left(\frac{1}{2}\right)^2+2\left(\frac{1}{2}\right)+1} = -2$

Therefore, $-2 \le y \le \frac{2}{3}$

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